Goal: degenerati

Explicit: $\begin{array}{c} W \\ \downarrow \\ C \end{array}$

$$GW(W_t) = \text{"}GW(W_0)\text{"} = \text{"}GW^{rel}(Y_1, D)\text{"} * \text{"}GW^{rel}(Y_2, D)\text{"}.$$

History: 1. Gauge-theory. Donaldson-Floer theory.

2. In algebraic geometry: D. Gieseker degeneration of moduli space

J. Harris. admissible covers.

3. In GW-theory, analytic approach, L-R. I-P.

E-H-G.

· Motivation: ——.

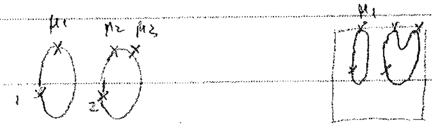
Relative stable morphisms

$D \subset Z$ stable maps $f: C \to Z$ with prescribed contact $f(C) \cap D$.

$\Gamma = g, n, (\mu_1, \cdots, \mu_m) = \mu$. $g(C) = g$

n - ordinary marked points. p_i

m - distinguished marked points. q_i



Require:

1) μ_i is the contact order of f at q_i

i.e. $f^*(D) = \sum \mu_i q_i$.

2) f is stable.

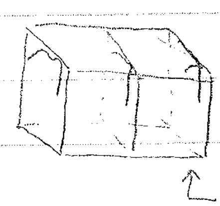
$Aut(f) = \{1\}$.

$M_\Gamma(Z^{rel}) = \{ f: C \to Z \mid \cdots \}$.

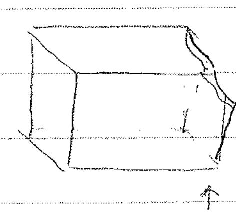
direct check: $M_\Gamma(Z^{rel})$ is DM —. Has perfect obstr. theory.

$M_\Gamma(Z^{rel})$ may be not proper.

Ideal: If we get a degenerate relative stable morphism

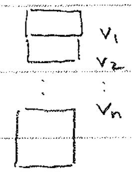


un acceptable specialization



the desired specialization.

Moral: $Z[n]_0$ =



elements in $M_\Gamma(X^{rel})$ are $f: C \to Z[n]_0$

s.t.:

1) C has genus g, ...

2) $f^*(D[n]_0) = \sum \mu_i q_i$.

3) f is stable, i.e. $Aut(f) = \{1\}$.

4) If $f(x) \in V_i$, then x is a node of C.

C_1, C_2 $\quad f_1^*(V_i) = \alpha_x x \quad f_2^*(V_i) = \alpha_x x$.

$M_\Gamma(X^{rel})$ as a D-M stack. (pre-deformable).

$Z[0] = Z$, $D[0] = D$.

$Z[n] = bl_{0 \times D[n-1]} \mathbb{A}^1 \times Z[n-1]$, $D[n]$ the proper transform of $D[n-1]$.

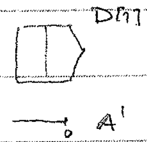
Key: $(\mathbb{C}^*)^n \circlearrowright (Z[n], D[n])$.

induced action on $Z[n]_0$ is exactly the $(\mathbb{C}^*)^n \circlearrowright Z[n]_0$

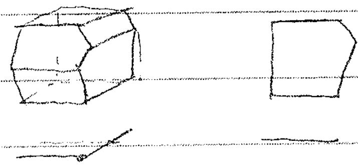
(easy: $f: C \to Z[n]_0$ is relative stable morphism, then

$n \leq M(g, \mu, n, d)$.)

$Z[1]$



$Z[2]$.



$Z[2] \supset Z[1]$ $\mathbb{C}^*$ equivariant.

Form moduli of relative stable morphisms ($M_\Gamma(Z[n]^{rel})$).

1. $M_\Gamma(Z[n]^{rel})^\circ \circlearrowleft (\mathbb{C}^*)^n$ finite stabilizer

2. $M_\Gamma(Z[n]^{rel})^\circ$ is a DM stack.

So $M_\Gamma(Z[n]^{rel})^\circ / (\mathbb{C}^*)^n$ is a DM stack.

3. $M_\Gamma(Z[n]^{rel})^\circ / (\mathbb{C}^*)^n \longrightarrow M_\Gamma(Z^{rel})$ is étale.

4. $M_\Gamma(Z^{rel})$ is separated and proper.

5. $M_\Gamma(Z^{rel})$ admits perfect-obstruction theory.

$$GW_\Gamma(Z^{rel}) : H^*(Z)^{\times n} \times H^*(M_\Gamma) \longrightarrow H_*(D)^{\times m}.$$

Degeneration formula

$$W \qquad W_0 = Y_1 \cup Y_2, \qquad D.$$
$$\downarrow$$
$$C$$

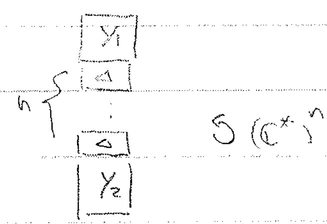
$C = \mathbb{A}^1$. construct $W[n] \circlearrowleft (\mathbb{C}^*)^n$ $\qquad \mathcal{W} = \lim W[n]$ as Artin stack.

$$\downarrow$$
$$\mathbb{A}^{n+1} \circlearrowleft (\mathbb{C}^*)^n$$

Degeneration of the moduli of stable morphisms:

$$\mathcal{M}_{g,n}(W,d) =_{set} \{ f: C \to W_t \mid f \text{ stable}, \dots \}_{t \neq 0}$$

$$\cup \{ f: C \to W[n]_0 \mid f \text{ stable}, \dots \} /\sim$$

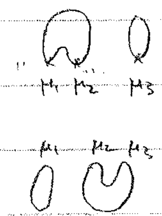


Theorem: 1. $\mathcal{M}_{g,n}(W,d)$ is naturally a DM stack

2. $\mathcal{M}_{g,n}(W,d)$ is sep & proper

3. $\mathcal{M}_{g,n}(W,d)$ admits a perfect obstruction complex.

How to derive a degeneration formula.



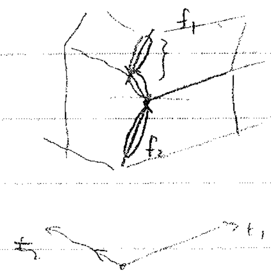
Γ_1

$(\Gamma_1, \Gamma_2, I)_\mu = (g, n, d)$.

Γ_2

$(f_1, f_2) \in \mathcal{M}_{\Gamma_1}(Y_1^{rel}) \times_{D^m} \mathcal{M}_{\Gamma_2}(Y_2^{rel}) \overset{\Phi_\eta}{\hookrightarrow} \mathcal{M}_\Gamma(W) \times_{\mathbb{C}} 0$.

immersion.



t_μ s.t. sing $\{t_\mu = 0\} \subset W[n]$

$(''t_2)$ is the nodal divisor where f_1 and f_2 are glued.

Any possible decomposition of $(g, n, d) = (\Gamma_1, \Gamma_2, I)_\mu$, there is an associated line bundle $\mathbb{L}_\mu$ on $\mathcal{M}_\Gamma(W)$ and a section t_μ.

s.t.

1. $t = \prod_\mu t_\mu$, $\quad \otimes \mathbb{L}_\mu' = \mathbb{1}$

2. $Z(t_\mu)$ homeo to $\mathcal{M}_{\Gamma_1}(Y_1^{rel}) \times_{D^m} \mathcal{M}_{\Gamma_2}(Y_2^{rel})$.

$$[\mathcal{M}_{g,n}(W_t, d)]^{vir} \simeq_{alg} [\mathcal{M}_\Gamma(W_0)]^{vir} = c_1(\mathbb{1}, t)\, [\mathcal{M}_\Gamma(W)]^{vir}$$

$$= \sum_\mu c_1(\mathbb{L}_\mu, t_\mu)\, [\mathcal{M}_\Gamma(W)]^{vir}$$

$Eq(\eta)$ Aut Γ_1, Γ_2

$$= \sum_\mu \frac{m(\mu)}{|Eq\,\eta|} \Phi_{\eta *}\, \Delta^! \left([\mathcal{M}(Y_1^{rel}, \Gamma_1)] \times [\mathcal{M}(Y_2^{rel}, \Gamma_2)] \right).$$