Rahul Pandharipande 03-12-2002

joint with A. Okounkov

Want to study the enumerative geometry of maps to a fixed smooth curve of genus g / $\mathbb{C}$.

There are two flavors of the theory:

- Gromov-Witten theory
- Hurwitz theory

and there is a precise way to go between those theories.

Plan: (i) GW theory of a target curve $X / \mathbb{C}$

(ii) Hurwitz theory

(iii) Completed cycles

(iv) GW / H correspondence.

(i) GW theory

Fix a smooth complex curve X. Consider

$$\overline{\mathcal{M}}_{g,n}(X,d) = \text{moduli of degree } d$$

maps from a curve of genus g with n-marked points to X.

We would be looking for formulas expressing

$$\langle T_{\alpha_1}(\gamma_1) \cdots T_{\alpha_n}(\gamma_n) \rangle^X_{g,d} :=$$

$$:= \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{vir}} \prod ev_i^*(\gamma_i)\, \psi_i^{\alpha_i}$$

Here:

- $ev_1, \ldots, ev_n : \overline{\mathcal{M}}_{g,n}(X,d) \to X$

are the evaluation maps at the marked points

$$ev_i^*(\gamma_i) \in H^*(\overline{\mathcal{M}}_{g,n}(X,d))$$

for $\gamma_i \in H^*(X)$.

- $\mathcal{L}_i \longrightarrow \overline{\mathcal{M}}_{g,n}(X,d)$ - line bundle

s.t.

$$(\mathcal{L}_i)_{(C \xrightarrow{f} X)} := (T^*C)_{p_i}$$

$$\psi_i = c_1(\mathcal{L}_i)$$

In our setup we will mostly work with

$$\gamma_i = [\text{point}] \in H^2(X)$$
$$X = \mathbb{P}^1$$

<u>Note</u>: We will work with $\overline{\mathcal{M}}_{g,n}^{\bullet}(X, d)$ - the moduli stack of maps with possibly disconnected domain. Since we have a dimension constraint we can drop g from the formulas. We will write

$$\langle \tau_{\alpha_1}(p) \cdots \tau_{\alpha_n}(p) \rangle^{\bullet}_d$$

for the corresponding GW invariants

(ii) <u>Hurwitz theory</u> $d \geq 1$

Choose $\theta_1, \dots, \theta_n$ - partition of d

$H^X_{g,d}(\theta_1, \dots, \theta_n)$ = # covers of genus g and degree d with ramification indices $\theta_1, \dots, \theta_n$ at the ramification points.

We will also have

$$H^{\bullet X}_{g,d}(\theta_1, \dots, \theta_n) = \# \text{ of possibly disconnected covers.}$$

To clear a slight mismatch between the two theories (one is defined for $d > 0$ the other is defined for $d = 0$.) note that again we can erase g from the Hurwitz numbers since it is redundant

We will write

$$H^{\bullet X}_{d}(\theta_1, \dots, \theta_n)$$

and we will extend the definition to all $d \geq 0$ and all partitions by saying

- If $d = 0$ the only non-zero Hurwitz number is

$$H^{\bullet X}_{d}(\emptyset, \dots, \emptyset) = 1$$

- $d > 0$, $\forall i$ $|\theta_i| \leq d$

$$\theta_i^! = \theta_i \cup 1^{d - |\theta_i|}$$

$$H_d^{\bullet X}(\theta_1, \ldots, \theta_n) = \prod \binom{m_i'}{m_i} H_d^{\bullet X}(\theta_1', \ldots, \theta_n')$$

where $m_i' = \#$ of parts of $\theta_i' = 1$
$m_i = \#$ of parts of $\theta_i = 1$.

- If $\exists\, |\theta_i| > d \Rightarrow H = 0$.

(iii) Completed cycles

Shifted symmetric functions: Consider the symmetric group $S(n)$ on n-letters acting on the polynomial ring $\mathbb{Q}[\lambda_1, \ldots, \lambda_n]$ by requiring that $S(n)$ acts via permutations on
$$\{\lambda_i - i\}_{i=1}^{n}.$$

This action is called the shifted symmetric action. The corresponding ring of invariants (the ring of shifted symmetric functions) is denoted by
$$\mathbb{Q}[\lambda_1, \ldots, \lambda_n]^{*S(n)}.$$

There is a natural map
$$\mathbb{Q}[\lambda_1, \ldots, \lambda_n]^{*S(n)} \longrightarrow \mathbb{Q}[\lambda_1, \ldots, \lambda_{n-1}]^{*S(n-1)}$$

given by

$$f(\lambda_1, \dots, \lambda_n) \to f(\lambda_1, \dots, \lambda_{n-1}, 0)$$

Exercise: That $f(\lambda_1, \dots, \lambda_{n-1}, 0)$ is also shifted symmetric.

Using these maps define

$$\Lambda^* := \varprojlim \mathbb{Q}[\lambda_1, \dots, \lambda_n]^{*S(n)}$$

Remark: How can we write explicitly elements in Λ^*?

First attempt: Take

$$p_\ell := \sum_{i=1}^{\infty} \left(\lambda_i - i + \frac{1}{2}\right)^\ell$$

These are shifted symmetric but divergent, so we have to correct it

True definition:

$$p_\ell := \sum_{i=1}^{\infty} \left[\left(\lambda_i - i + \frac{1}{2}\right)^\ell - \left(-i + \frac{1}{2}\right)^\ell\right] + \ell!\, c_\ell$$

where $\{c_\ell\}$ are constants defined by

$$\sum_{j=0}^{\infty} c_j z^j = \frac{\frac{z}{2}}{\sinh\left(\frac{z}{2}\right)}$$

One can check that

$$\Lambda^* = \mathbb{Q}[p_1, p_2, \ldots]$$

In particular given a partition $\mu = \mu_1 \mu_2 \ldots, \mu_k$ we can define

$$p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_k}.$$

and $\{p_\mu\}_{\mu-\text{partition}}$ is a linear basis of Λ^* over $\mathbb{Q}$.

There is another basis of Λ^* indexed by partitions : $\{f_\mu\}_{\mu-\text{partition}}$

Let λ be a partition

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

Define

$$f_\mu(\lambda) = \begin{cases} \binom{|\lambda|}{|\mu|} |C_\mu| \dfrac{\chi_\mu^\lambda}{\dim \lambda} & \text{if } |\lambda| \geq |\mu| \\ 0 & \text{if } |\lambda| < |\mu| \end{cases}$$

Here χ^λ_μ are defined as follows

Consider

$C_\mu \subset S(|\mu|)$ — the conjugacy class of μ

Then

$$\begin{array}{ccccc} S(|\mu|) & \overset{i}{\hookrightarrow} & S(|\lambda|) & \xrightarrow{\chi^\lambda} & \mathbb{Q} \\ \cup & & & \nearrow_{\chi^\lambda_\mu} & \\ C_\mu & & & & \end{array}$$

<u>Thm</u> (Kerov–Olshanski)

(i) $f_\mu \in \Lambda^*$ for all μ

(ii) $\deg(f_\mu) = |\mu|$

(iii) $\{f_\mu\}$ is a linear basis of Λ^*

Consider

$$\phi : \bigoplus_{n=0}^{\infty} Z(\mathbb{Q}S(n)) \longrightarrow \Lambda^*$$

center of the group ring

given by

$$\phi(C_\mu) := f_\mu$$

Then define completed conjugacy class

$$\overline{C_\mu} := \phi^{-1}\left(\frac{p_\mu}{\mu_1 \cdots \mu_k}\right)$$

$$(\overline{\mu}) = \overline{C_\mu}$$

$$(\bar{\ell}) = \phi^{-1}\left(\frac{p_\ell}{\ell}\right) = \sum_{\mu,\ |\mu| \le \ell} \varepsilon(\mu)\, I(\mu, (\bar{\ell}))\cdot\mu$$

Here $\varepsilon(\mu) = \mathrm{Aut}(\mu) \cdot \prod \mu_1 \cdots \mu_n$

$(\bar{\ell}) = (\ell) + \cdots$ and $I(\mu, (\bar{\ell}))$ is the coefficient of μ.

<u>Examples:</u>

$$(\bar{1}) = (1) - \frac{1}{24}(\ )$$

empty partition

$$(\bar{2}) = (2)$$

$$(\bar{3}) = (3) + (1,1) + \frac{1}{12}(1) + \frac{7}{2880}(\ )$$

(iv) The GW/H correspondence

$$\tau_k(p) \longleftrightarrow \frac{\overline{(k+1)}}{k!}$$

establishes the GW/H correspondence.

More precisely

$$\langle \tau_{k_1}(p) \cdots \tau_{k_n}(p) \rangle_d^{\bullet X} = \frac{H_d^{\bullet X}\left(\overline{(k_1+1)}, \ldots, \overline{(k_n+1)}\right)}{k_1!\, k_2! \cdots k_n!}$$

Geometric motivation:

Since the completed cycles are corrected regular cycles $\Rightarrow$ have an expansion of RHS with leading term

$$\frac{H_d^{\bullet X}\left((k_1+1), \ldots, (k_n+1)\right)}{k_1! \cdots k_n!}$$

This term can be interpreted geometrically as follows: There is an open substance $\mathcal{M}_{g,n}(X,d) \subset \overline{\mathcal{M}}_{g,n}(X,d)$

consisting of maps with non-singular domains

Then

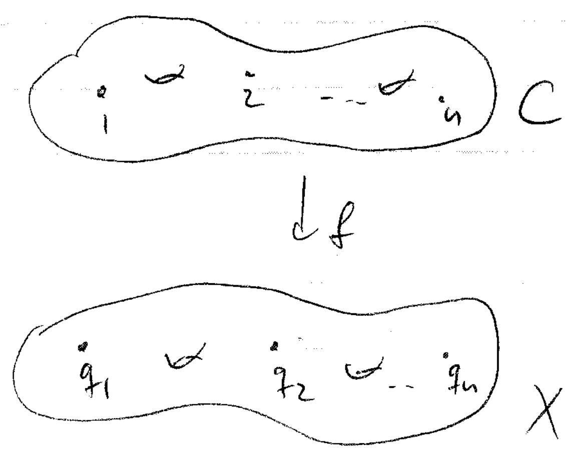
$$\prod \psi_i^{k_i}\, ev_i^*(p) \cap [\mathcal{M}_{g,n}(X,d)]$$

$$\parallel$$

$$\frac{1}{\prod_i k_i!}\left(\begin{array}{c}\text{0-cycle of maps counting} \\ H_d^{\bullet X}((k_1+1),\dots,(k_n+1))\end{array} \right)$$

Indeed

$$\prod ev_i^{-1}(q_i) \cap [\mathcal{M}_{g,n}(X,d)] \quad :$$



$$\Rightarrow \left(df_* \quad \underset{\substack{\uparrow \\ \text{fixed}}}{T^*_{q_1} X} \to T^*_1 C \right)$$

$\Rightarrow$ corresponds to a section

s_1 of $\mathcal{L}_1$

$$\text{Zero}(s_1) = \text{all } (C \xrightarrow{f} X) \text{ s.t. } f \text{ is ramified at } 1.$$

$\Rightarrow$ $\psi_1 \longleftrightarrow$ locus of $(C \xrightarrow{f} X)$ s.t. f is ramified at 1

Now on $Z(s_1)$

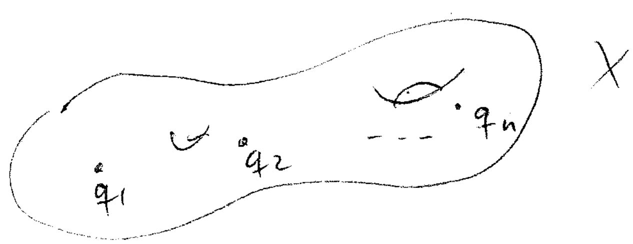
$$\frac{m_{q_1}}{m_{q_1}^2} \longrightarrow \frac{m_{z_1}^2}{m_{z_1}^3} \Rightarrow \text{section } s_2 \quad \mathbb{C} \to \mathcal{L}^{\otimes 2} \text{ on } Z(s_1)$$

$\Rightarrow$

$$\psi_1(2\psi_1) = \text{locus where } f \text{ is ramified } z \to z^3$$

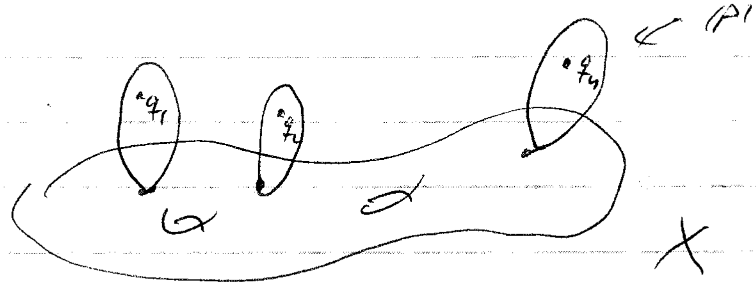
$\Rightarrow$ $\psi_1^2 = \frac{1}{2}(3)$, etc.

Degeneration formula:



Want to compute $\langle \tau_{\alpha_1}(q_1) \cdots \tau_{\alpha_n}(q_n) \rangle_d^{\bullet X}$.

To do this computation we can degenerate X to



and then compute. The answer is expressed in terms of relative GW invariants

$$\langle T_{\alpha_1}(q_1) \cdots T_{\alpha_n}(q_n) \rangle_d^{\bullet X} = \sum_{\theta_1, \dots, \theta_n \vdash d} \left(H_d^{\bullet X}(\theta_1, \dots, \theta_n) \cdot \prod_{i=1}^{n} \epsilon(\theta_i) \langle \theta_i, T_{\alpha_i}(p) \rangle_d^{\bullet \mathbb{P}^1} \right)$$

The connection with the general GW/H correspondence is given by the formula

$$\langle \mu, T_k(p) \rangle_d^{\mathbb{P}^1} = \frac{I(\mu, \overline{(k+1)})}{k!}$$

<u>Note</u>: On the LHS we have the <u>connected</u> GW invariants.

How can we compute $\langle \mu, T_k(p) \rangle_d^{\mathbb{P}^1}$?

Form a generating series

$$T_\mu(z) = \sum_{k=0}^{\infty} \langle \mu, T_k(p) \rangle_d^{|p|} \, z^{2g(\mu,k)}$$

It can be computed explicitly from the representation theory of the symmetric group.

Define

$$\mathcal{S}(z) := \frac{\sinh\left(\frac{z}{2}\right)}{\left(\frac{z}{2}\right)}$$

Then

$$T_\mu(z) = \prod_{i=1}^{\text{length}\,\mu} \frac{\mathcal{S}(\mu_i z)}{|\mathrm{Aut}(\mu)|} \cdot \frac{\mathcal{S}(z)^{|\mu|-1}}{|\mu|!}$$

where

$$\mu = \{\mu_1, \ldots, \mu_{\text{length}\,\mu}\}.$$

The proof of this will be given in Okounkov's talk.

<u>Burnside Formula</u>:

$$H_d^{\bullet |p|}(\theta_1, \ldots, \theta_n) = \sum_{|\lambda| = d} \left(\frac{\dim \lambda}{d!}\right)^2 \prod_{i=1}^{n} f_{\theta_i}(\lambda).$$

Now by the GW/H correspondence we get

$$\langle \tau_{k_1}(p) \cdots \tau_{k_n}(p) \rangle_d^{\bullet \mathbb{P}^1} = \sum_{|\lambda| = d} \left(\frac{\dim \lambda}{d!} \right)^2 \cdot$$

$$\cdot \prod_{i=1}^{n} \left(\frac{\mathbf{p}_{k_i+1}(\lambda)}{(k_i+1)!} \right)$$