Multigraded Hilbert Schemes

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Q: How to describe all d-dim'l subspaces of $V \simeq k^n$?

A: Grassmannian $Gr(d, k^n)$

Similar for a graded vector space $V = \bigoplus_{a \in A} V_a$ and $d : A \to \mathbb{N}$

Q: How to describe all ideals I in $S = k[x_1, \dots, x_n]$
$= \bigoplus_{a \in A} S_a$ with Hilbert function $h : A \to \mathbb{N}$?

$\dim_k(S_a/I_a) = h(a)$

A: The Hilbert scheme H_S^h exists

Four Examples

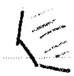
① Negative degrees

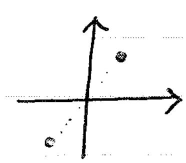
$n = 2$, $A = \mathbb{Z}$, $\deg(x) = 1$, $\deg(y) = -1$, $h = \mathbb{1}$

$I = \langle xy - \alpha \rangle$, H_S^h = the affine line $= \mathbb{A}_k^1$

② Orbits of finite abelian groups

$n = 2$, $A = \mathbb{Z}/2\mathbb{Z}$, $\deg(x) = \deg(y) = 1$, $h = \mathbb{1}$

$I = \langle x^2 - \alpha, \beta_0 x - \beta_1 y \rangle$, $H_S^h = T^* \mathbb{P}_k^1$ 

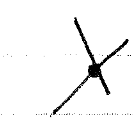


③ The smallest reducible Hilbert scheme

$n = 3$, $A = \mathbb{Z}^2$, $\deg(x) = (1,0)$, $\deg(y) = (1,1)$, $\deg(z) = (0,1)$, $h =$

	1	1
1	2	1
1	1	1

$I = \langle x^3, x^2y, xy^2, y^3, y^2z, z^2, \alpha_0 x^2 z - \alpha_1 xy, \beta_0 xyz - \beta_1 y^2 \rangle$

$H_S^h = \{(\alpha, \beta) \in \mathbb{P}^1 \times \mathbb{P}^1 : \alpha_1 \beta_1 = 0\}$ = two $\mathbb{P}^1$'s glued at a point 

④ Every ideal has its own Hilbert scheme

$n = 6$, $I = \langle ae - bd, af - cd, bf - ce \rangle \subset k\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

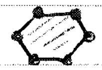
($\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ or $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ or $\Delta \subseteq$ [illegible]

~~Riddle: What is the dimension of the zeroset of I~~ ?

~~Answer: Four (in $\mathbb{C}^2$), three (in $\mathbb{P}^5$), two (in $\mathbb{P}^2 \times \mathbb{P}^2$), one (in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$)~~, zero ☺

$A \simeq \mathbb{Z}^4$: the finest grading which makes I homogeneous, $h = h_I$

$H_S^h = \mathbb{C}^6/(\mathbb{C}^*)^4$ scaling rows and columns

$=$  blow-up of $\mathbb{P}^2$ at three points

$=$ the space of triangulations of  $= \mathrm{conv}\{\deg(a), \ldots, \deg(f)\}$

__Theorem__: The toric Hilbert scheme ($h = \mathbb{1}$) has a natural morphism to the Chow quotient $\mathbb{C}^n/G$, where $G = \mathrm{Hom}(A, \mathbb{C}^*)$

__Theorem__ (Santos 2002): Both spaces can be disconnected.

Equations Defining Hilbert Schemes

__Maclagan's Lemma__: Antichains of monomial ideals are finite

A finite subset $D \subset A$ is __supportive__ for $h: A \to \mathbb{N}$ if

(g) Every monomial ideal with Hilbert function h is generated in degrees D.

(h) Every monomial ideal I generated in degrees D satisfies $\forall a \in D: h_I(a) = h(a) \Rightarrow \forall a \in A: h_I(a) \leq h(a)$

__Theorem__ If D is supportive then the restriction morphism $H_S^h \to H_{S_D}^h$ is a closed embedding, defined by the __natural determinantal equations__.

We call $D \subset A$ __very supportive__ for $h: A \to \mathbb{N}$ if

(g), (g'): first syzygies, (h'): $=$ instead of $\leq$

__Theorem__ If D is very supportive then $H_S^h \simeq H_{S_D}^h$ and is defined by the __natural quadratic equations__ (provided $S_0 = k$)

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Grothendieck's Hilbert Scheme

$$\mathcal{H}ilb_p = \{ \text{subschemes of } \mathbb{P}^{n-1} \text{ with Hilbert polynomial } p \}$$

Let $d_0 = d_0(p,n)$ be the Gotzmann number

Fix $A = \mathbb{Z}$, $\deg(x_i) = 1$ and $h(a) = \begin{cases} \binom{n+a-1}{a} & \text{if } a < d_0 \\ p(a) & \text{if } a > d_0 \end{cases}$

<u>Theorem</u> (Gotzmann) $\mathcal{H}ilb_p = H_S^h$, $\{d_0\}$ is supportive and $\{d_0, d_0+1\}$ is very supportive

<u>Grothendieck's embedding</u> is given by the natural determinantal equations

$$\mathcal{H}ilb_p \hookrightarrow Gr(p(d_0), S_{d_0})$$

[redacted]

<u>Gotzmann's embedding</u> is given by the natural quadratic equations

$$\mathcal{H}ilb_p \hookrightarrow Gr(p(d_0), S_{d_0}) \times Gr(p(d_0+1), S_{d_0+1})$$

<u>Theorem</u>: Bayer's 1982 Conjecture is true:
His equations of degree n define [redacted] scheme-theoretically.