Chern-Simons theories in dimensions $4, 5, 6$

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Chern-Simons theories in dimension $4, 5, 6$ are non-topological field theories. There has been a great deal of work on these theories in the last few years:

1. $4d$ CS is a unifying framework for integrable $2d$ models
2. $5d$ CS is a supersymmetric sector of M-theory – gives an accessible model of holography
3. $6d$ CS is related to some "integrable" $4d$ QFTs (and also holography)

I will survey some of these developments, mostly focused on a nice geometric relationship between

$$4d \text{ CS} \Longleftrightarrow 2d \text{ integrable PDEs/ integrable QFTs}$$

I will survey work by:

Jacob Abajian, K.C., Francois Delduc, Richard Derryberry, Davide Gaiotto, Sylvain Lacroix, Si Li, Marc Magro, Jihwan Oh, Natalie Paquette, Benoit Vicedo, Brian Williams, Edward Witten, Masahito Yamazaki, Yehao Zhou, and others

What is Chern-Simons theory in dimension > 3?

Prototype

Holomorphic Chern-Simons: X a Calabi-Yau 3-fold, $A \in \Omega^{0,1}(X, \mathfrak{g})$ a $\overline{\partial}$ connection,

$$hCS(A) = \int_X \Omega_X \wedge CS(A).$$

Equations of motion:

$$F^{(0,2)}(A) = 0$$

A defines a holomorphic bundle.

$4d$ CS: dimensional reduction

$$\mathbb{C}^\times \times \mathbb{C}^\times \times \Sigma \rightsquigarrow \mathbb{R}^2 \times \Sigma \tag{1}$$

4d Chern-Simons: consider the 4 manifold $\Sigma_1 \times \Sigma_2$ where Σ_i are Riemann surfaces and ω is a meromorphic one-form on Σ_2 with no zeroes.

Gauge field:

$$A \in \Omega^1(\Sigma_1 \times \Sigma_2) \text{ modulo } \Omega^{1,0}(\Sigma_2)$$

a gauge field with with no $(1,0)$ term in Σ_2 direction.

Lagrangian:

$$\int_{\mathbb{R}^2 \times \Sigma} \omega \wedge CS(A)$$

Equations of motion:

$$\omega F(A) = 0$$

Holomorphic bundle on Σ_2, flat bundle on Σ_1, in a compatible way.

Integrable PDEs

We will see that 4d CS gives a unified understanding of 2d integrable PDEs/integrable field theories.

Basic example of an integrable PDE: G a compact Lie group, $\sigma : \mathbb{R}^2 \to G$ a map.

Harmonic map equation on σ is an integrable PDE.

Lax presentation: From σ we can build $\nabla(\sigma, z)$ a principal G-bundle with connection on $\mathbb{R}^2$ depending meromorphically on spectral parameter $z \in \mathbb{CP}^1$ such that

$$F(\nabla(\sigma, z)) = 0 \text{ for all } z \iff \sigma \text{ is harmonic.}$$

Formula for $\nabla(\sigma, z)$:

$$\nabla^{1,0} = \sigma^{-1}\partial\sigma\frac{1}{1-z}$$
$$\nabla^{0,1} = \sigma^{-1}\overline{\partial}\sigma\frac{1}{1+z}$$

Important generalization: include WZW term. Euler-Lagrange equations for

$$\int_{\Sigma}\langle \mathrm{d}\sigma, *\mathrm{d}\sigma\rangle_G + c\int_{M^3}\widehat{\sigma}^* MC \tag{2}$$

$\mathrm{d}M^3 = \Sigma$, $MC \in \Omega^3(G)$ is Maurer-Cartan 3-form.

This remains an integrable PDE for all values of c.

Integrability and conserved quantities

Lax formulation of integrability implies there are infinitely many conserved quantities.

View σ as a map $\mathbb{R} \times S^1 \to G$, or as $\mathbb{R} \to LG$.

Define

$$M(z,t) = \mathrm{Hol}_{t \times S^1}(\nabla(z,\sigma))$$

Function of σ, harmonic:

$$M(z,t) \in C^\infty(T^* LG)$$

Conserved quantity:

$$\partial_t M(z,t) = 0$$

from flatness of $\nabla(z,\sigma)$.

(Also $\{M(z), M(z')\} = 0$).

Given Riemannian manifold (M, g) with closed 3-form Ω, when is harmonic map equation on (M, g, Ω) integrable?

Fairly small list of traditional examples:

1. G as above.
2. Riemannian symmetric spaces.
3. Certain deformations of these: e.g. Fateev-Onofri-Zamolodchikov sausage, S^2 with metric

$$\frac{e^t - e^{-t}}{e^t + e^{-t} + e^{-2x} + e^{2x}} (\mathrm{d}x^2 + \mathrm{d}\theta^2) \tag{3}$$

(t a parameter, (x, θ) coordinates)

Integrable PDEs from 4*d* CS

Consider 4*d* Chern-Simons on $\mathbb{R}^2 \times \Sigma$, ω meromorphic one-form on Σ.

Assume ω has

1. Double poles
2. Simple zeroes

We will find EOM of 4*d* CS on $\mathbb{R}^2 \times \Sigma$ can be rewritten as maps

$$\sigma : \mathbb{R}^2 \to \mathcal{M}(\Sigma, G) \tag{4}$$

where $\mathcal{M}(\Sigma, G)$ has metric g, three-form Ω. Harmonic map equation is automatically integrable!

$\Sigma = \mathbb{CP}^1$: recovers all previously known examples.

Poles and zeroes

Local coordinate z on Σ, complex coordinate w on $\mathbb{R}^2$.

When ω has a pole

$$\int CS(A)\frac{\mathrm{d}z}{z^2}$$

is not gauge invariant. Solution: require that $A = 0$ at $z = 0$.

When ω has a zero

$$\int CS(A) z \mathrm{d}z = \int A_w \partial_{\overline{z}} A_{\overline{w}} z \mathrm{d}\overline{z} \mathrm{d}z + \dots \tag{5}$$

is *not elliptic*.

Solution: ask that A_w (or $A_{\overline{w}}$) has a first order pole in z. (n poles, $2g - 2 + 2n$ zeroes: $g - 1 + n$ have $A_w = 0$, $g - 1 + n$ have $A_{\overline{w}} = 0$).

Assume Σ is defined over $\mathbb{R}$, all zeroes/poles of ω are real, choose a real form of G.

Let $\mathcal{M}(\Sigma, \omega)$ be the moduli of semistable G bundles on Σ over $\mathbb{R}$, trivialized at poles of ω.

Theorem

Equations of motion of 4dCS are equivalent to harmonic map equation for a map

$$\mathbb{R}^2 \to \mathcal{M}(\Sigma, \omega) \tag{6}$$

$\mathcal{M}(\Sigma, \omega)$ *has a canonically defined real-algebraic metric and 3-form, built from the Szëgo kernel defined using* ω.

Theorem (Conjectured by C. and Yamazaki, proved by R. Derryberry)

The harmonic map equation with target $\mathcal{M}(\Sigma, \omega)$ is always integrable.

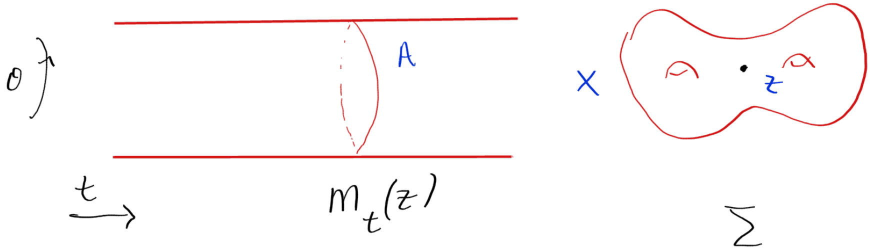
Proof.

Direct proof due to R. Derryberry – he explicitly computes, without reference to $4d$ CS, that flatness of $\nabla(z, \sigma)$ is equivalent to the harmonic map equation. □

General $4d$ CS argument: Lax connection $\nabla(z, \sigma)$ comes from gauge field A in $4d$ CS, flatness of $\nabla(z, \sigma)$ comes from equations of motion of $4d$ CS

$$F_{w\overline{w}}(A) = 0.$$

Monodromy matrix $M(z)$ comes from holonomy of the gauge field in 4d CS.



Conservation of $M(z)$ follows from $4d$ CS equations of motion $\omega F(A) = 0$.

Traditional examples: $\Sigma = \mathbb{CP}^1$.

$$\omega = \mathrm{d}z \frac{(z - p_1)(z - p_2)}{z^2} \tag{7}$$

Poles at $z = 0$, $z = \infty$, zeroes at p_1, p_2.

$\mathcal{M}(\Sigma, \omega)$ is G.

Metric is $(p_1 - p_2)\kappa$, three form is $(p_1 + p_2)MC$.

$$\omega = \mathrm{d}z \frac{(z - p_1)\dots(z - p_{2n})}{(z - q_1)^2 \dots (z - q_n)^2} \tag{8}$$

$n + 1$ poles, $2n$ zeroes, $\mathcal{M}(\Sigma, \omega) = G^n$.

Symmetric spaces/FOZ sausage metric, etc : modifications of the construction – first order poles in ω, branch cuts,...

$g > 0$: very hard to write an explicit global form of the metric. It is defined on each tangent space using the Szëgo kernel.

Ricci flow and integrability

Riemannian manifold with closed 3-form: can apply modified Ricci flow, variation of g depends on 3-form:

$$\delta g_{\mu\nu} = \mathrm{Ric}_{\mu\nu} - \frac{1}{4}\Omega_{\mu\zeta\eta}\Omega_{\nu}^{\zeta\eta}$$

Ricci flow is closely tied with integrability: e.g. FOZ sausage metric is the unique ancient solution to Ricci flow equation in dimension 2.

Conjecture

The family of manifolds $\mathcal{M}(\Sigma, \omega)$ is closed under modified Ricci flow. Further, Ricci flow corresponds to an explicit geometric flow on the moduli space of (Σ, ω)

Geometric flow and Ricci flow

(Σ, ω) has n second order poles, $2g - 2 + 2n$ simple zeroes, divided into groups p_i^+, p_i^- of size $g - 1 + n$.

There is a natural flow on the moduli of (Σ, ω) so that

$$\delta \int_{p_i^-}^{p_j^+} \omega = 1 \quad \delta \oint \omega = 0 \quad \delta \int_{p_i^\pm}^{p_j^\pm} = 0. \tag{9}$$

(Construction of flow : remove small neighbourhood of p_i^+, glue back in with singular vector field $\frac{1}{\omega}$).

Conjecture

This flow is proportional to the modified Ricci flow.

True in genus 0 (Delduc, Lacroix, Magro, Vicedo) .

$5d$ non-commutative Chern-Simons

On $\mathbb{R} \times \mathbb{C}^2$ can write a gauge field

$$A \in \Omega^1(\mathbb{R} \times \mathbb{C}^2, \mathfrak{gl}_n)/\Omega^{1,0}(\mathbb{C}^2) \tag{10}$$

with Lagrangian

$$\int \mathrm{d}z_1 \mathrm{d}z_2 \,\mathrm{tr}(A \mathrm{d}A) + \tfrac{2}{3}\,\mathrm{tr}(A * A * A) \tag{11}$$

where $*$ combines $\wedge$ and Moyal product:

$$A_1 * A_2 = A_1 \wedge A_2 + c\epsilon^{ij}\partial_{z_i} A_1 \wedge \partial_{z_j} A + \dots \tag{12}$$

c a formal variable.

Non-commutativity seems to be essential to build quantum theory.

5d non-commutative CS is a super-symmetric sector of M-theory:

M-theory on

$$(\mathbb{R}^2_{\epsilon_1} \times \mathbb{R}^2_{\epsilon_2})/\mathbb{Z}_{K-1} \times \mathbb{R}^2_{-\epsilon_1-\epsilon_2} \times \mathbb{R} \times \mathbb{C}^2$$

is 5d non-commutative CS for $\mathfrak{gl}_K$.

Conjectured by K.C., largely proved by Richard Eager, Fabian Hahner, Surya Raghavendran and Brian Williams.

ϵ_i: equivariant parameters ("Ω-background").

Holography: can match supersymmetric OPEs on N $M2$ branes (or $M5$ branes) with computations in $5d$ non-commutative CS [1]

N $M2$ branes in this setting: QM particle moving on ADHM moduli space of rank K instantons on $\mathbb{R}^4$ of charge N.

Theorem

The following three algebras are equal:

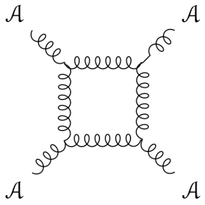
1. *$M2$ algebra of operators as $N \to \infty$*
2. *A certain quantization $U_\hbar(\mathfrak{gl}_K \otimes \mathrm{Diff}(\mathbb{C}))$ ("shifted affine Yangian")*
3. *Koszul dual of algebra of operators of $5d$ CS*

[1] K.C., Abajian-Gaiotto, Oh-Zhou, Gaiotto-Rapčak, related to Mezei-Pufu-Wang

Holomorphic Chern-Simons

Most natural variant is holomorphic Chern-Simons on a Calabi-Yau 3-fold: $\mathcal{A} \in \Omega^{0,1}(X)$, $\int \Omega_X \wedge CS(\mathcal{A})$.

Problem: this does not exist as a quantum theory because of a gauge anomaly.



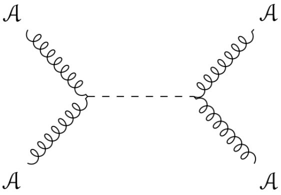
Math terms: X a CY3,

$$c_1(\mathrm{Bun}_G(X)) = \int_X \mathrm{Td}(TX)\,\mathrm{ch}(\mathrm{Ad}_{\mathfrak{g}})$$

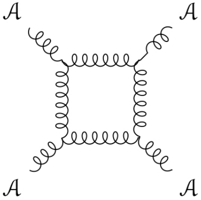
(Grothendieck-Hirzebruch-Riemann-Roch). $\mathrm{Td}_0(TX)\,\mathrm{ch}_4(\mathrm{Ad}_{\mathfrak{g}})$ is non-vanishing even if X is flat.

Solution[2] : include Kodaira-Spencer field[3] $\mu \in \Omega^{0,1}(X, TX)$.

Anomaly cancels by Green-Schwarz mechanism when $G = SO(8)$, $G_2 \times G_2$:



cancels with



if

$$\mathrm{Tr}_{adjoint}(X^4) \propto \mathrm{tr}(X^2)^2 \tag{13}$$

$X \in \mathfrak{g}$; also need $\dim \mathfrak{g} = 28$.

[2]K.C., Si Li

[3]Bershadsky, Cecotti, Ooguri, Vafa

Why is holomorphic CS interesting?

1. Mirror symmetry (mirror to counts of un-oriented curves)
2. Construction of $4d$ integrable field theories.

Holomorphic CS plus Kodaira-Spencer theory, placed on twistor space

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1 \cong \mathbb{R}^4 \times \mathbb{CP}^1$$

gives rise to a very interesting QFT on $\mathbb{R}^4$.

Field

$$\sigma : \mathbb{R}^4 \to SO(8)$$

and Kähler potential ρ. Lagrangian

$$\int_{\mathbb{R}^4} \mathrm{Tr}\, J \wedge *J + \frac{1}{3} \int_{\mathbb{R}^4} (\alpha + \partial\rho) \wedge \mathrm{Tr}(J \wedge [J, J]) + \dots$$

$\mathrm{d}\alpha = \omega$, Kähler form.

1. Power-counting non-renormalizable
2. Even so, defined uniquely at the quantum level!
3. Forced to include gravity (by Green-Schwarz mechanism).
4. Has strong hints of integrability.