Random walks on weakly hyperbolic groups

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Random and Arithmetic Structures in Topology
MSRI - Fall 2020

Random walks on weakly hyperbolic groups - Summary

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Random walks, WPD actions, and the Cremona group

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Answer. It depends on the topography (geometry) of the city.

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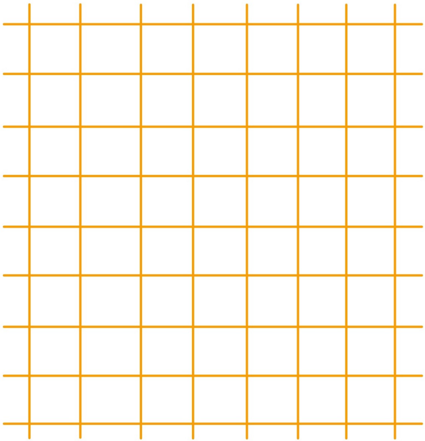

What is the probability of coming back to where you started?

Recurrence

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Exercise. Prove the Lemma.

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$\therefore$ our RW is recurrent.

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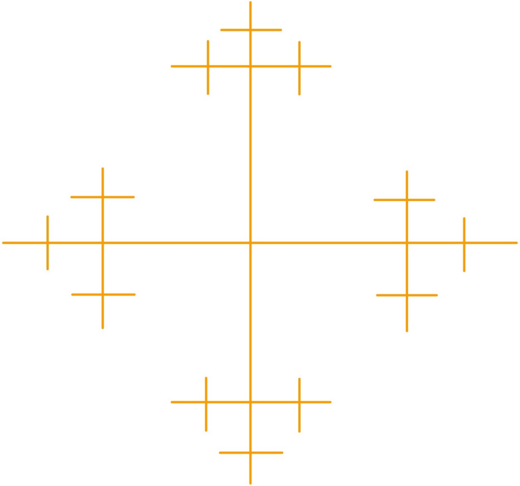
Exercise. Prove Polya's theorem for $d = 3$. Moreover, for the simple random walk on $\mathbb{Z}^d$, show that $p^{2n}(0,0) \approx n^{-\frac{d}{2}}$.

Transient random walks

Example 2: Tree City

In Tree City, the map has the shape of a 4-valent tree.



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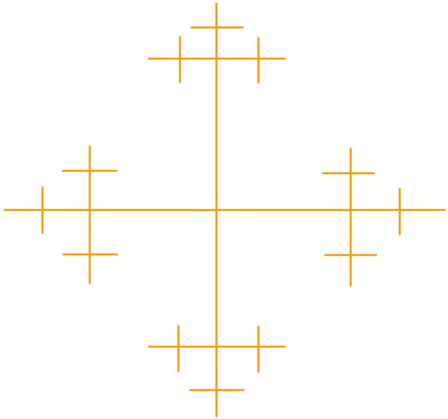

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(do we know $\lim_{n\to\infty} \frac{d_n}{n}$ exist?)

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If you fix a basepoint $x \in X$ you can look at the sequence $(w_n \cdot x) \subseteq X$.

Examples

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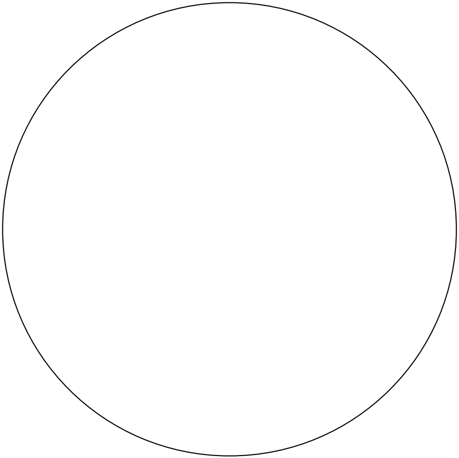

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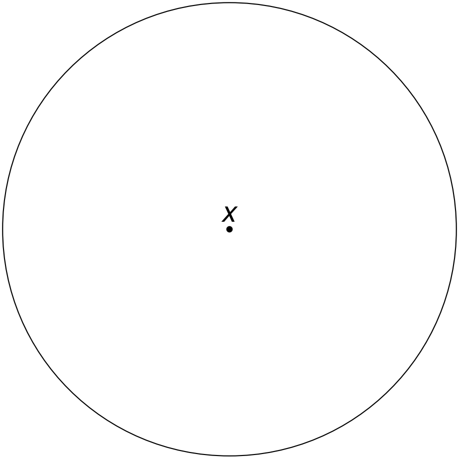

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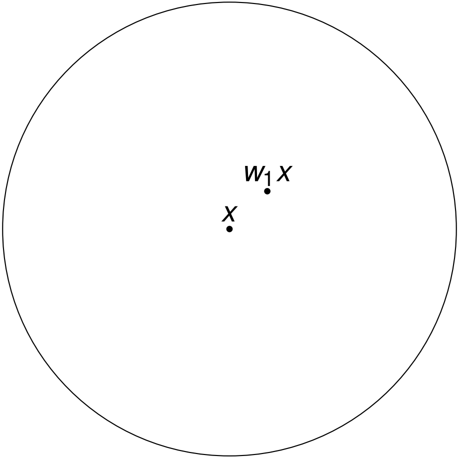

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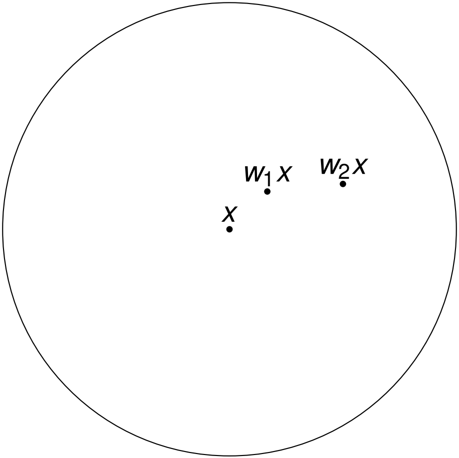

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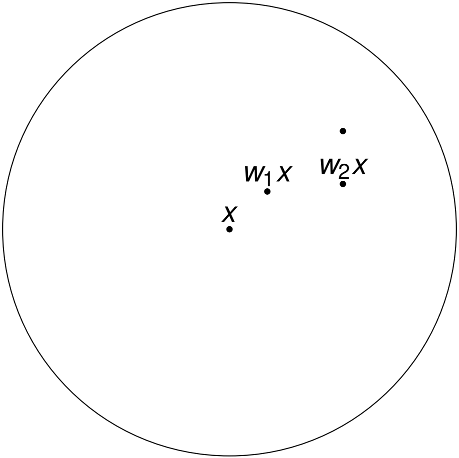

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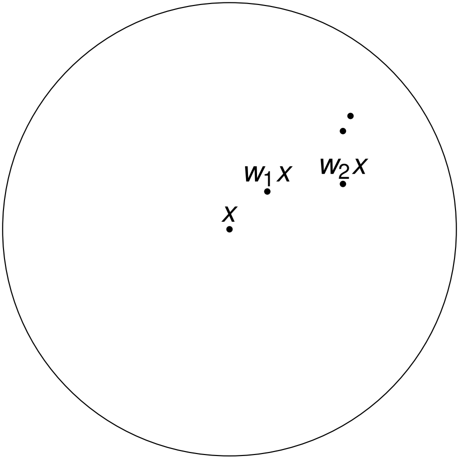

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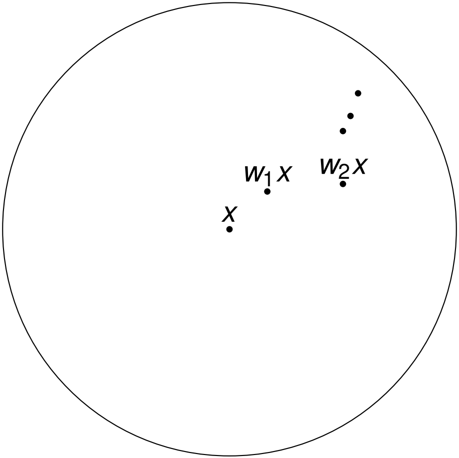

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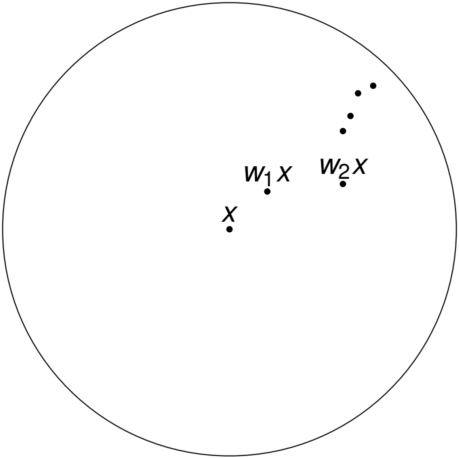

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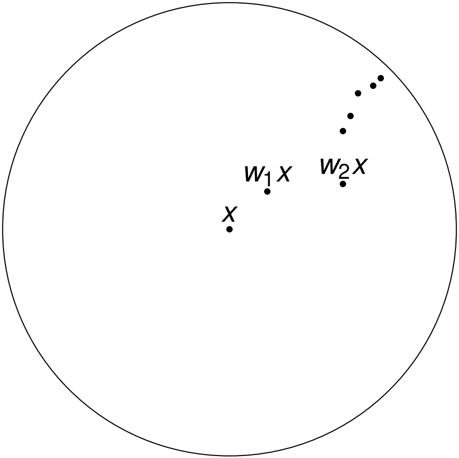

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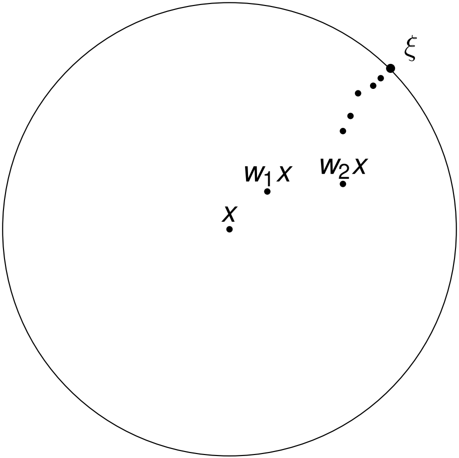

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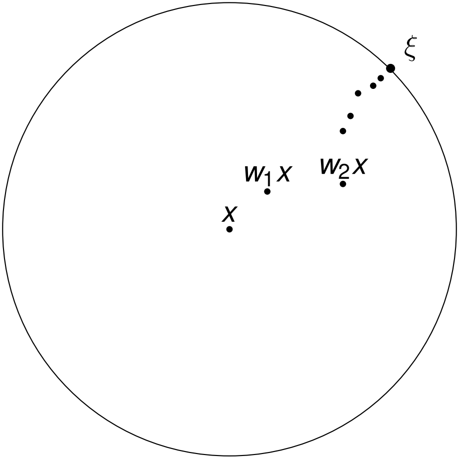

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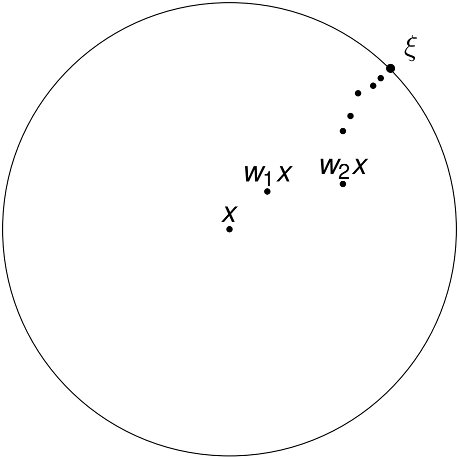

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Lemma

If μ has finite first moment, then there exists a constant $L \in \mathbb{R}$ such that for a.e. sample path

$$\lim_{n\to\infty} \frac{d(w_n x, x)}{n} = L.$$

Proof.

For any $x \in X$, the function $a(n,\omega) := d(x, w_n(\omega)x)$ is a subadditive cocycle, because

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6. Is $(\partial X, \nu)$ a model for the Poisson boundary of (G, μ)? That is, do you have a representation formula for bounded harmonic functions?

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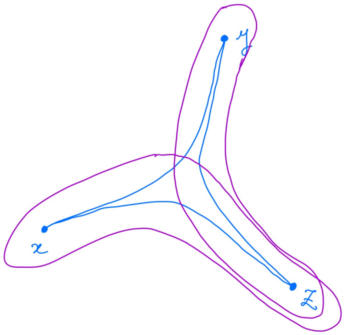

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Recall a space is proper if metric balls $\{z \in X \ : \ d(x, z) \leq R\}$ are compact.

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