Mathematical Sciences Research Institute

Final Report on the
Mathematical Sciences Research Institute
2020 African Diaspora Joint Mathematics Workshops (ADJOINT)
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Mathematical Sciences Research Institute

2020 African Diaspora Joint Mathematics Workshop (ADJOINT) Report

The **African Diaspora Joint Mathematics Workshop (ADJOINT)** is a two-week summer activity designed to provide opportunities for in-person research collaboration to U.S. mathematicians, especially those from the African Diaspora, who work in small groups with research leaders on various research projects. The program includes professional development activities and ongoing support for the groups to continue their research collaborations beyond the summer through periodic virtual meetings with collaborators.

Through this program, MSRI aims to establish and promote research communities that will foster and strengthen research productivity and career development among its participants. The ADJOINT workshops are designed to catalyze research collaborations, increase the visibility of its researchers by facilitating their participation in and organization of conferences and to develop a sense of community among the participating mathematicians. This program will enhance the mathematical sciences and its community by positively affecting the research and careers of African-American mathematicians.

In 2020, because of the COVID-19 pandemic, ADJOINT's 2 week summer activities took place virtually from June 15 to 25, 2020. Below is a summary of the program's structure, participants, impact, evaluations, and recommendations for the program.

Structure and Participants:

The ADJOINT 2020 program took place virtually (due to the COVID-19 pandemic) from June 15 to 26 during the summer of 2020. A total of 22 researchers (including five Research Leaders) participated in one of the five working groups. The groups were comprised of mathematicians, predominantly of African descent, at various career stages, and the five Research Leaders, accomplished and renowned researchers in their respective fields, were also predominantly of African descent. The 2020 ADJOINT Research Leaders included Tepper Gill (Howard University), Abba Gumel (Arizona State University), Ryan Hynd

(University of Pennsylvania), Bonita V. Saunders (National Institute of Standards and Technology), and Craig Sutton (Dartmouth College). Their research projects were in the areas of mathematical physics, mathematical biology, analysis and PDEs, statistics, and differential geometry. See Table 2 for a complete list of research groups and participants.

Each participant completed a pre-program survey to assess their level of research activity coming into the program and what they hoped to gain from their participation. Also, each participant, research leader and the on-site leader was provided with an iPad equipped with applications and software to support the technical needs of each research team and facilitate collaboration from remote locations. In advance of the program, each research team received a training session with one of the members of the MSRI IT staff to introduce the iPad's capabilities.

The two week program was structured around synchronous research collaboration time for the five research groups. The program began with a synchronous opening session with all participants, where MSRI leadership, ADJOINT directors and MSRI support staff were also present. At this time, all participants and research leaders were introduced and the program schedule was outlined. From there, participants began working in their individual research groups, with a dedicated Zoom link for each research team, for the remainder of the program. Accounting for the fact that participants' time spanned different time zones, at least six hours per day of the program was reserved for participants to be working synchronously within their individual research groups with their research leader. During the first week, time was reserved in the morning for research leaders to present to their groups, whereas the remainder of the day was expected to consist of more collaborative work. The schedule included a shared lunch break where participants were encouraged to join the large group to socialize and network with other participants on various topics of conversation. The schedule also included three professional development workshops on grant-writing, tenure and promotion and leadership/marketability. The workshops were structured as panel discussions with invited speakers (including both external guests and some program participants with expertise in the topic) providing guidance, advice and mentoring to the participants. Finally, on each Friday of the two week program, all five research teams gave 15--20 presentations on the progress of their research.

Table 2: List of Research Groups from the 2020 ADJOINT program

Research Topic	Participants	Institution
Analysis, PDEs, and Mathematical Physics	Tepper Gill (Leader) Eddy Kwessi Douglas Mupasiri Timothy Myers	Howard University Trinity University University of Northern Iowa Howard University
Mathematics of the Transmission Dynamics and Control of the 2019 Novel Coronavirus	Abba Gumel (Leader) Kamal Barley Keisha Cook Sherry Scott Lawrence Udeigwe Talitha Washington	Arizona State University Stony Brook University Tulane University Milwaukee School of Engineering Manhattan College Howard University / National Science Foundation
Hamilton-Jacobi equations in high dimensions	Ryan Hynd (Leader) Romeo Awi Dennis Ikpe Henok Mawi Terrance Pendleton	University of Pennsylvania Hampton University Michigan State University Howard University Drake University
Validated Numerical Computations of Mathematical Functions	Bonita V. Saunders (Leader) Rachel Vincent-Finley Sean Brooks Ron Buckmire	National Institute of Standards and Technology Southern University and A&M College Coppin State University Occidental College
Explorations in Inverse Spectral Geometry	Craig Sutton (Leader) Lotfi Hermi Donald King	Dartmouth College Florida International University Dartmouth College

Short and Long-term Impacts on the Participants:

The impact of ADJOINT 2020 on participants is reflected positively in the post-program evaluations, which suggest that participants' overall were very satisfied with their research experience, the guidance of their research leader, their interactions with other participants and the support of the program staff. In terms of short-term impacts, participants met a diverse group of African-American researchers and professionals in the mathematical sciences. Each research group presented twice to the large group on the progress of their research so that participants became familiar with all the research teams and leaders and their areas of interest and expertise. This encouraged participants to interact with each other across research groups and make potential research connections with other research teams. The professional development workshops provided another venue for participants to network with other professionals who can provide support and guidance on their present and future career trajectories. One particularly fortunate impact that resulted from the virtual nature of the program was that participants became practiced in new technologies for remote collaboration and communication in the mathematical sciences. Their technology experience will further support the expectation for continued collaboration after the conclusion of the program. In terms of long-term impacts, program participants are now part of a larger network of African-American researchers, despite the fact that many will return after the program to institutions where they may feel isolated or part of a very small minority. The continued research collaboration beyond the program will provide long-term engagement with this wider community of African-American researchers and future opportunities to be featured in venues beyond the ADJOINT program.

Continued Collaboration, Research and Career Development Activities after the Initial 2 Week Summer Activities:

Since the two-week summer workshop, there have been two subsequent program events designed to increase visibility and productivity of program participants. First, the ADJOINT program planned two research seminars to be hosted at MSRI. The first seminar was held at the end of the fall semester (December 9, 2020) and the second is scheduled for the end of the spring semester (April 27, 2021). These seminar events are designed to feature 2020 program participants as the guest speakers and include a social hour before each lecture. Again due to the ongoing COVID-19 pandemic, these seminar events were designed to be held virtually on Zoom. The fall seminar featured Dr. Sherry Scott as the guest speaker and the spring seminar will feature Drs. Rachel Vincent-Finley and Sean Brooks in a joint talk. The turnout for the fall seminar was high and the event was well-received by the participants. In fact, many participants stayed beyond the scheduled time in order to socialize and interact further. The turnout for the fall seminar indicates that the research community established in the summer of 2020 continues to grow. We are hoping for a high turnout for the spring seminar as well. Second, the ADJOINT program directors organized a special session at the 2021 Joint Mathematics Meeting. The special session featured 13 speakers, all of whom were 2019 or 2020 program participants or research leaders. The list of speakers in the ADJOINT special session at JMM can be found at:

https://www.jointmathematicsmeetings.org/meetings/national/jmm2021/2247_program_ss84.html#title.

We have also proposed a four-hour special session for ADJOINT at the 2022 Joint Mathematics Meeting.

Publications:

At the time of this report, four manuscripts from 2020 ADJOINT researchers have been submitted for publication and a fifth one has been published. The manuscript titles are listed below:

Submitted for Publication:

1. "On the Equivalence Between Weak BMO Spaces and the Space of Derivatives of the Zygmund Class," Eddy Kwessi.
2. "Two Critical Times for the SIR model," Ryan Hynd, Dennis Ikpe, Terrance Pendleton.
3. "Continuous Time Approximation of Nash Equilibria," Ryan Hynd, Romeo Awi, Henok Mawi.
4. "An Eradication Time Problem for the SIR model," Ryan Hynd, Dennis Ikpe, Terrance Pendleton.

Published:

1. "Postlockdown Dynamics of COVID-19 in New York, Florida, Arizona, and Wisconsin," Keisha Cook, Sherry E. Scott, Kamal Barley.

Impact on Research Leaders and Directors:

The impact of the program on the research leaders is reflected positively in post-program evaluations and activities. Four of the five research leaders rated the program as "very helpful" or "extremely helpful" in improving their connection to the community of African American mathematical research scientists. Four of the five also reported the research experience at ADJOINT "very likely" or "extremely likely" to develop into a publication.

The leadership and research expertise of the research leaders also impacted other participants through their group's research presentations. Some research leaders even reported that they would be "very likely" to initiate research collaborations with participants outside of their group.

Overall, the research leaders have been very supportive and generous to the ADJOINT program by offering their time, expertise and experience when called upon, well beyond the two-week workshop. For instance, the research leaders provided feedback to the program to inform the planning of the ADJOINT seminar in December and the JMM special session in January, and they attended a meeting in April with the four 2021 Research Leaders and 2021 on-site director where they spoke very highly of the program and enthusiastically provided guidance and advice for the benefit of 2021 summer workshop.

The research leaders have also been instrumental in providing the ADJOINT directors with guidance about support for their research teams, but moreover, their participation in the program as a research leader is expected to be a positive addition to their professional leadership record and experience. As an example, after MSRI became aware of Dr. Abba Gumel's research, as a result of the ADJOINT 2020 program, Dr. Gumel was asked to join the Organizing committee for MSRI's Hot Topic workshop on Mathematical Models for Prediction and Control of Epidemics. Given the high sensitivity of this subject during the COVID 19 pandemic, the workshop was extremely well attended. More than 500 mathematical and statistical scientists listened to talks given by world experts. Dr. Gumel's talk was very well received and the large audience provided him with an exceptional audience. Subsequently, Dr. Gumel gave an invited Current Events talk at the 2021 Joint Mathematics Meetings. Other ADJOINT research leaders have also been featured prominently in national meetings since the summer 2020 program: Dr. Ryan Hynd gave an AMS Invited Address at the 2021 Joint Mathematics Meetings, Dr. Tepper Gill gave an invited talk at the AMS-NAM Joint Special Session on Celebrating the Mathematical Legacy of Dr. James A. Donaldson at the 2021 Joint Mathematics Meeting, and Dr. Bonita Saunders will give the AWM-MAA Etta Zuber Falconer Lecture at the 2021 MAA MATHFEST. These examples demonstrate the importance of visibility for researchers in the mathematical sciences overall, but particularly for those from underrepresented groups.

Finally, the increased visibility applies not only to the Research Leaders but also to the ADJOINT Program Directors. By interacting frequently with the MSRI directors and staff, the program directors have broadened their network within the mathematical community as well and this increased visibility may facilitate opportunities to serve as potential collaborators, leaders and organizers for other programs, organizations or committees.

Cross-Program Interactions and Mentoring - ADJOINT and MSRI-UP:

During the summer 2020 ADJOINT workshop, participants were invited to a meet and greet with MSRI-UP program. The meet and greet, coordinated by the MSRI-UP and ADJOINT on-site directors Dr. Duane Cooper and Dr. Naiomi Cameron, took place on the last day of the ADJOINT program (June 26, 2020). All of the MSRI-UP students and several of the ADJOINT participants and research leaders attended this meet and greet. The ADJOINT participants each introduced themselves to the students and spoke briefly about the research projects. In addition, the ADJOINT participants were able to share a bit about their personal journeys, professional activities and share some professional advice and/or opportunities with students. The MSRI-UP students were able to further interact with the ADJOINT Research Leaders and participants who were present during an informal Q&A session.

Research Group Reports:

Each research leader was requested to provide a 1--2 page written report of the two week program experience by July 26, 2020. The report was expected to summarize the research experience during the two week program as well as outline the goals and plans for the research collaboration going forward. Research leaders were encouraged to indicate in the report any plans for maintaining the group's communication, any anticipated outcomes, such as conference presentations, publications, grant proposals, etc., or any requests for additional support. The reports will be used to inform the follow up communication with research groups regarding their progress.

We received full reports from four of the five research leaders in the 2020 program. One report is outstanding. See Appendix I for the 2020 ADJOINT Research Teams' reports. While the reports we received from the 2020 research leaders were informative and helpful, there was some variation in the format, scope and detail of the reports. Hence, for the summer 2021 program, we intend to implement a template and provide more guidelines for the report. The information gathered from the reports will be used as a starting point to track the progress of each research group over time.

Summary of Exit Surveys & Evaluation:

Participants of both the 2019 (pilot) and 2020 MSRI ADJOINT workshops were asked to complete an exit survey of their experience at the end of the on-site program. The 2020 ADJOINT group completed a pre-survey as well. After the 2020 workshop, we retained a consultant, Karen Peterman Consulting, Co., to conduct interviews with past program participants and to provide a follow-up assessment report based on the interviews with 2019 and 2020 cohorts and an analysis of the pre/post-surveys.

The exit surveys collected from the ADJOINT 2020 program were very positive. About 94% of ADJOINT participants reported that the program has been or will be helpful in achieving in increasing their long-term research output (papers, presentations, etc.), improving their professional network and initiating or strengthening connections within the African American mathematical sciences community. About 88% of program participants reported that ADJOINT has been or will be helpful in collaborating with people whom they would not otherwise have access to, and 82% said the program has or will be helpful in learning a new research area. These responses are in alignment with the goals and objectives of the program.

While the overwhelming majority of the feedback on the effectiveness of the research groups was positive, there was some indication from the surveys that one of the research groups was not as robust as the other four. Though the surveys were intentionally anonymous, it is clear that one research leader rated the likelihood of outcomes such as publications and conference presentations slightly lower than the other four. We suspected that this difference may have been related to the research group with the smallest size and, perhaps consequently, the largest variance in degree of expertise in the subject area. This is something we kept in mind as we went through the selection and matching process for applicants and research leaders for the 2021 and 2022 programs, as well as through our communication and guidance to potential participants. For a variety of reasons, for the 2021 program, we settled on four research groups. One result of having four research groups in 2021 (versus five in 2020) is that the research teams will be more balanced in size. We will continue to monitor group size as a potential factor in the effectiveness of the group collaboration.

Here is a sample of some of the positive comments provided by participants in their surveys.

- *I had an excellent time! And we are still meeting as a group even after the workshop ended. I expect that our group will continue on...thank you!*
- *The fact that we could communicate with each other freely without fear of being judged. Our relationships were completely free of the racial undertones that characterize every aspect of what we do in our workplaces. This I think is one of the factors that made the last two weeks so much more productive.*
- *The aspects of the relationships with colleagues that are most beneficial to me include (1) learning about the paths that colleagues have taken; (2) learning about the participation of colleagues outside of academia; (3) learning about how others have navigated their professional journey; (4) gaining insight and advise concerning my next steps; (5) learning about outreach and teaching initiatives have been implemented at other institutions.*
- *This is an [extremely] program, and MSRI is ahead of the curve, when it comes to the politics of science, and can push matters in the right direction, nationally and internationally.*
- *This is a much needed and timely program. If we have learned anything at all in the last few weeks it is that the uneven distribution of opportunities among races is a big problem. Math is one of the most segregated fields of study. That has to change and African American mathematics researchers can play a role in making that change.*

While many of the comments on the 2020 post-survey were positive, we realize that it is important to have a thoughtful retrospective analysis in order to design the most useful, proactive and sustainable evaluation of the program going forward. Thus, Karen Peterman's assessment report has provided further insight for consideration of the overall program goals and outcomes. Based on cohort interviews and surveys, her report proposes eight recommendations which the program directors are actively considering as we plan for the 2021 and 2022 workshops and subsequent activities:

- *continue to provide structure to program activities that support opportunities for group learning among all members of the cohort, as well as devoted time for meeting as research teams*
- *continue to improve and expand outreach efforts to ensure that Black faculty in the mathematical and statistical sciences are aware of ADJOINT and the benefits it offers to participants and research leads*
- *consider the ways in which programming can continue to serve and leverage the strengths among program participants across different stages of their careers*
- *identify opportunities for former ADJOINT participants to stay engaged in ADJOINT programming, either through joining new cohorts of scholars as a research participant or through intentional capacity-building to participate as a research lead or co-lead in the future*
- *continue to identify professional development activities that are of most interest to program participants at specific career stages*
- *strengthen training opportunities for ADJOINT participants to become a change agent in the mathematical sciences (if ADJOINT continues to see itself operating in this space toward this goal)*
- *strengthen ADJOINT'S mentoring infrastructure by (1) assigning mentors to junior faculty who participate in the program, and (2) providing space for discussion of racial prejudice and discrimination within the field and academia*
- *consider administering the program survey using a retrospective-pre survey design (i.e., where the respondent answers questions related to their knowledge, attitudes, and behaviors before and after participating in the program in one sitting after completion of the program)*

While some of these recommendations are already underway, the program directors are considering additional ways that these recommendations can be implemented into the programming for 2021 and 2022.

APPENDIX:

RESEARCH GROUP REPORTS

Final Report
2020 African Diaspora Joint Mathematics Workshop
Analysis, PDE's and Mathematical Physics
June 15-26, 2020

Submitted by:
Tepper L. Gill[1]
Research Leader

Submitted to:
The Office of the Director
Mathematical Sciences Research Institute
Berkeley, California

Participants:
Eddy Kwessi[2], Douglas Mupassiri[3] and Timothy Myers[4]

[1] Departments of EECS and Mathematics
Howard University, Wash. DC
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Trinity University, San Antonio TX.
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Howard University, Wash. D

Introduction

The analysis, PDE's and mathematical physics program focused on analysis and PDE's because of the interest and backgrounds of the participants. Dr Kwessi was interested in analysis and Dr Gill's research on the spaces of functions of weak bounded variation, $BMO^{w}[\mathbf{R}^{n}]$ and their relationship to the Hardy space, $H^{1}[\mathbf{R}^{n}]$; Dr Mupasiri was interested in Dr Gill's research in operator theory and geometry of Banach spaces; while Dr Myers was interested in Dr Gill's research on integration on infinite-dimensional spaces and applications to PDE's.

In late April, the program director sent three papers to the participants broadly covering the general background material related to their interests. The program proper began with a set of three basic lectures covering operator theory on Banach spaces, the construction and properties of the spaces of functions for non-absolutely integrable functions and the construction and study of Lebesgue measure on $\mathbf{R}^{\infty}$.

Program Plan

The first meeting began with formal introductions by the participants and Professor Gill provided and sought agreement on the following program plan. Dr Gill would present three lectures on the three topics proposed by the participants. During the research time, each participant would provide a lecture on his current work as it related to his proposal. This would ensure that all were familiar with each other's research area. The day after each presentation, professor Gill refined his lecture to provide the latest known work and problems that may be studied in that area. It was agreed that the remainder of the time would be spent in attempting to provide answers to the questions and problems that arose.

Lectures and Problems for Further Study

During the first lecture, it was shown that that every separable Banach space B, can be continuously and densely embedded into a unique separable Hilbert space H. This led to the expectation that, by projection, some equivalent structures from the Hilbert H will be transferred into the Banach space B. This led to two questions:

1. What structures from the Hilbert space are not transferred into the Banach space?
2. Can every non-separable Banach space be continuously and densely embedded into a unique non-separable Hilbert space?

Professor Mupasiri discussed the first question and presented his ideas on a number of interesting research directions. The first was to see if his research on shape preserving projections for Banach spaces could be lifted to Hilbert space via the embedding. The second led to some deep questions about the implicit assumption that, the use of algebraic and topological ideas and tools in functional analysis did not cause any hidden problems.

The second question was raised by Professor Kwessi after Dr Gill noted that the space BMO was not separable. Dr Gill later presented a special presentation on the space BOM^w and showed that it contained BMO as a continuous dense embedding. This led to a refinement of Dr Kwessi's questions to:

1. What is the dual of BMO^w and
2. Does there exist a Hilbert space H, which contains BMO^w.

The actual interest centered on discovering the dual of BMO^w. Fefferman had shown that the dual of BMO is H^1. Dr Kwessi conjectured that the dual of BMO^w might be the space constructed by DeSousa.

During the second lecture, a new class of separable Banach spaces $KS^p[\mathbf{R}^n]$ were constructed for $1 \le p \le \infty$. These spaces contain all functions that have a bounded integral. This includes the special and wide-sense integrals of Denjoy. The special Denjoy integral is equivalent to the Perron and the Henstock-Kurzweil integral, while the wide-sense Denjoy integral is equivalent to the Khinchin-integral. The Hilbert space $KS^2[\mathbf{R}^n]$ was use to provide the first rigorous development of the Feynman formulation of quantum mechanics. The following theorem encapsulates the important properties of these spaces:

Theorem For $1 \le p \le \infty$:

1. A weakly compact set in $L^p[\mathbf{R}^n]$, is strongly compact in $KS^p[\mathbf{R}^n]$.
2. For $1 < p < \infty$, $KS^p[\mathbf{R}^n]$ is uniformly convex.
3. For $\frac{1}{p} + \frac{1}{q} = 1$, $KS^p[\mathbf{R}^n]$ is the dual space of $KS^q[\mathbf{R}^n]$.

4. The space of test functions $\mathcal{D}[\mathbf{R}^n]$ are contained in $KS^p[\mathbf{R}^n]$ as a continuous dense embedding.

(The condition 4. implies that $KS^q[\mathbf{R}^n]$ contains the space of distributions.)

The class of functions of weak bounded variation, $BMO^w[\mathbf{R}^n]$, $BMO^{-w}[\mathbf{R}^n]$ and the class of Zachary functions of weak bounded variation, $Z^p[\mathbf{R}^n]$, $Z^{-p}[\mathbf{R}^n]$ were also introduced. These are also spaces that contain all the functions with bounded integrals.

During the third lecture, it was shown that Lebesgue measure could be constructed on $\mathbf{R}^\infty$ in a way that is very close to the way one constructs it on $\mathbf{R}^n$. Professor Myers conjectured that the dual use of the embedding and the construction of Lebesgue measure might allow him to extend the theory of Fourier transforms to every separable Banach space. This would lead to a constructive theory of PDE's on separable Banach spaces.

Future Plans and Outcomes

The second week of the program was devoted to refining all planned research activities. The following outcomes and plans were agreed upon:

1. Each participant expects to have at least one research paper submitted for publication by the end of the summer.
2. A proposal was written for a special session on advances in analysis and PDE's at the 2021 Mathematics Congress of the America's.
3. During the month of July, a one to two-hour weekly zoom meeting by Dr Gill with each participant to discuss problems and progress was planned and agreed on.

Report Submitted by Abba Gumel, Research Leader
Mathematics of the Transmission Dynamics and Control of the 2019 Novel Coronavirus

ADJOINT 2020: Report of Gumel's Group

The working group addressed the most important public health and socio-economic challenge facing mankind over the last 102 years, namely the problem of the spread of the novel 2019 Coronavirus (COVID-19). The group consists of Kamal Barley (Stony Brook University), Keisha Cook (Tulane University),). Abba Gumel (Arizona State University), Sherry Scott (Milwaukee School of Engineering) and Lawrence Udeigwe (Manhattan College).

We started with a general discussion on what each group member expected to gain from the ADJOINT 2020 program. The responses included trying to submit a paper to a reputable Mathematical Biology journal, exploring collaboration opportunities withing our group and the rest of the 2020 ADJOINT cohort, learn more about mathematical epidemiology, learn about funding opportunities in mathematical biology and how to write winning research proposals.

The group leader gave a series of lectures introducing the basic principles of mathematical biology. In particular, the lectures covered the history of mathematical modeling of infectious diseases (dating back to the pioneering works of Daniel Bernoulli, Sir Ronald Ross, Kermack and McKendrick etc.). We started with the fundamental compartmental modeling framework of the Kermack-McKendrick formulation of SIR (susceptible-infected-recovered) epidemic models, including detailed discussion on the fundamental assumptions (such as homogeneous mixing, exponentially-distributed waiting times in epidemiological compartments, no vital/demographic processes, to account for the assumption that the time scale of the epidemic is less than the human demographic timescale, large population size etc.). We discussed the derivation of basic qualitative features of the SIR model, including proving the non-negativity and boundedness of solutions, invariance of solutions, computation of reproduction numbers, conditions for the existence and asymptotic stability of solutions and characterizing bifurcation types.

The lectures also covered extended versions of the classical Kermack-McKendrick SIR model, such as models with an exposed class (SEIR), demographic dynamics, multiple disease stages and vaccination (and the computation of vaccine-derived herd immunity threshold). Various techniques for computing the reproduction number of disease transmission models, notably inspection method, standard linearization method and the next generation operator method, were discussed. We also covered ways to design effective control strategies based on the expression of the associated reproduction number of the model. The lectures also covered the important concept of parameter estimation, data fitting and uncertainty quantification/analysis. The group leader also presented some of his earlier papers on COVID-19. In addition to the series of lectures, the group also studied some of the fundamental research papers in mathematical biology, including the 2000 SIAM Review by Hethcote (titled ``Mathematics of Infectious Diseases"). The group also read some of the recommended texts on Dynamical Systems, to refresh their background on the theory and methodologies of nonlinear dynamical systems (in particular, the books by Strogatz and Perko were quite useful to the group).

After acquiring the basic background knowledge, we were then ready to start addressing our actual project. We started with the formulation of a basic model for the spread of COVID-19 in the United States. The model was designed by splitting the total population into mutually-exclusive compartments based on disease status. Some of the notable features of the basic model included the incorporation of a compartment for asymptomatically-infectious individuals, as well as allowing for the assessment of numerous non-pharmaceutical interventions, such as the use of face masks, testing and contact-tracing of confirmed cases, self-isolation etc.   The group carried out extensive literature search to find data and parameter values needed to fit (and parameterized) the model. The group also carried out a detailed qualitative analysis of the model, including the computation of its reproduction number and proving the global asymptotic stability property of the continuum of disease-free equilibria, using Lyapunov function theory and the LaSalle's Invariance Principle. The group's code for fitting the data did not work well, and we decided to explore that after the two-week period.   The group presented their work (to the entire ADJOINT 2020 cohort) during the first and second weeks of the workshop.

The group identified some projects pertinent to COVID-19 dynamics to explore in the future. These include studying the impact of COVID-19 on healthcare workers, exploring the impact of a potential anti-COVID-19 vaccine and adding stochasticity (associated with modeling superspreading events) into models for COVID-19 dynamics. The group members plan to continue meeting online periodically to work on the project. The group members will be making presentations of their work in conferences, and plan to write a research manuscript or more.

Report

Subgroup 1: Approximating Nash Equilibria
Subgroup 2: Time Optimal Control for Compartmental Models in Epidemiology

Submitted by:
Ryan Hynd
Research Leader

Participants:
Subgroup 1: Romeo Awi, Henok Mawi

Subgroup 2: Dennis Ikpe, Terrance Pendleton

Report

My group split into two subgroups. One subgroup (myself, Romeo Awi, and Henok Mawi) is working on approximating Nash equilibria. The other (myself, Dennis Ikpe, and Terrance Pendelton) is working on time optimal control for compartmental models in epidemiology. I'll describe roughly what the specific projects involve below.

Approximating Nash equilibria. The notion of a Nash equilibrium is an important concept in the theory of noncooperative games. It is informally described as a collective strategy assumed by several players in which no player can decrease her cost by changing her individual strategy. In terms of mathematics, the collective strategy set can be modeled as a cartesian product

$$X := X_1 \times \cdots \times X_N,$$

where each X_i represents the possible strategies of the ith player. If $f_1, \dots, f_N : X \to \mathbb{R}$ represent the cost functions of the respective players, $x \in X$ is a *Nash equilibrium* provided

$$f_i(x) \leq f_i(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_N)$$

for all $y_i \in X_i$ and $i = 1, \dots, N$.

Under appropriate continuity and compactness assumptions, it is known that there is a Nash equilibrium. This is typically proved with an analog of Brouwer's fixed point theorem. As a result, this existence result is nonconstructive. Nevertheless, we are interested in identifying conditions on $f_1, \dots, f_N$ and X such that there is a Nash equilibrium which can be approximated by a constructive method. When each X_i is a subset of $\mathbb{R}^d$ (or a Hilbert space), our method involves the large time limit of solutions of the differential inclusions

$$\begin{cases} \dot{\gamma}_j(t) \in -\partial_{x_j} f_j(\gamma(t)), \text{ a.e. } t \in (0, \infty) \\ \gamma_j(0) = y_j. \end{cases} \tag{0.1}$$

We are working on establishing the existence and uniqueness of solutions of (0.1) and in understanding precisely when $\gamma(t) = (\gamma_1(t), \dots, \gamma_N(t))$ converges to a Nash equilibrium as $t \to \infty$.

Time optimal control for the SIR model. The SIR model is perhaps the best known epidemiology model. It predicts how three compartments of the total population evolve in time. We consider a variant of this model which involves a vaccination rate

$$r : [0, \infty) \to [0, 1].$$

Namely, we study the ODE

$$\begin{cases} \dot{S} = -\beta S I - r S \\ \dot{I} = \beta S I - \gamma I \\ \dot{R} = \gamma I \end{cases} \tag{0.2}$$

where $S, I, R : [0, \infty) \to \mathbb{R}$ represent the susceptible, infected, and recovered compartments of a total population of size N. Here β and γ are the respective infected and recovery rates per unit time and r represents a controlled vaccination rate.

The problem we have been studying is to characterize the vaccination rate which minimizes the time in which the infected population falls below a given threshold

$$\mu > 0.$$

That is, we seek to characterize r such that minimizes the eradication time

$$\min\{t \geq 0 : I^r(t) \leq \mu\}.$$

Here S^r, I^r, R^r is the solution of the (0.2) a given set of initial conditions. And if we allow the initial conditions

$$S^r(0) = x \quad \text{and} \quad I^r(0) = y > \mu$$

to vary, we can obtain a solution $u(x, y)$ of the Hamilton-Jacobi-Bellman equation

$$\beta xy\partial_x u + x(\partial_x u)^+ + (\gamma y - \beta xy)\partial_y u = 1. \tag{0.3}$$

Moreover, this solution promises to be a useful tool in helping us solve our problem of time optimal control. As a result, we have been studying solutions of (0.3) and the various inequalities they satisfy.

African Diaspora Joint Mathematics Workshop (ADJOINT) 2020

Report of

Validated Numerical Computations of Mathematical Functions Research Group

Research Leader: Bonita V. Saunders, Applied and Computational Mathematics Division (ACMD),

National Institute of Standards and Technology (NIST)

Group Participants: Sean Brooks (Coppin State University), Ron Buckmire (Occidental College), Rachel Vincent-Finley (Southern University and A&M College)

Project Background:

During the late 1930s, 40s and 50s accurate tables of function values were calculated by human 'computers' to facilitate the evaluation of functions by interpolation. In addition to logarithmic and trigonometric functions, these reference tables included values for gamma, Legendre, Jacobian, Bessel, Airy, and other high level or "special" functions arising in mathematical and physical applications.

The advent of reliable computing machines, computer algebra systems, and computational packages diminished the need for such reference tables, but today's researchers and software developers still need a way to confirm the accuracy of numerical codes that compute mathematical function values. The field of validated computations of special mathematical functions explores the development of codes that compute certifiably accurate function values that can be used to test the accuracy of values produced by personal, commercial, or publicly available codes.

Project Goals:

Provide group participants with an accessible introduction to the field of validated numerical computing by assisting them in learning the relevant terminology and techniques for error analysis. Study and evaluate the current implementation of validated computing in an ongoing collaborative effort between NIST and the University of Antwerp. Explore opportunities to continue the ADJOINT 2020 group's work and present and publish results in appropriate venues and journals.

Glossary:

ACMD - Applied and Computational Mathematics Division - part of the NIST Information Technology Laboratory (ITL). ACMD conducts research in the mathematical, computational, and physical sciences.

DLMF – NIST Digital Library of Mathematical Functions: https://dlmf.nist.gov . The DLMF is a complete re-write of the well cited (Abramowitz and Stegun, 1964) Handbook of Mathematical Functions published by the National Bureau of Standards (now NIST). Bonita Saunders managed the development of over 600 graphs and 200 interactive visualizations for the project and serves on its Editorial Board.

DLMF Tables – DLMF Standard Reference Tables on Demand: http://dlmftables.uantwerpen.be. A spin-off of the DLMF project, this is a collaborative effort between the NIST ACMD and Annie Cuyt's Computational Mathematics (CMA) Research Group at the University of Antwerp in Belgium to create an online system that outputs values of high level mathematical functions to user specified precision with an error bound to certify the value's accuracy. The project is being led by Bonita Saunders.

ADJOINT 2020 Accomplishments: While all participants were familiar with floating-point systems, absolute and relative errors, and other terms found in elementary numerical analysis courses, most were not familiar with the intricate tracking and analysis of rounding errors needed to create high precision codes for testing. However, as the group successfully delved into difficult references (Higham, 2002), (Backeljauw, 2014), (Cuyt, 2008), estimates of what could be accomplished evolved upward significantly. Together the group plowed through the Backeljauw paper, which explains the error analysis driving the computing engine of NIST DLMF Tables. Specific accomplishments:

- Learned the relevant terminology and computational methods for validated computing, including general principles of finite precision computation, floating point arithmetic, rounding errors, etc. For computational methods, the group focused on Taylor series and continued fractions, the primary methods used in DLMF Tables.
- Developed an understanding of the error analysis used to generate certified function values to user specified precision. Extensive reading of several Higham chapters and the Backeljauw paper.

Future Plans: The group will continue the work already started and connect with other members of the NIST and Antwerp groups that make up the NIST DLMF Tables team. Specific plans are provided below. The first two bullets and examining the C++ code are already in progress. Currently, the ADJOINT group plans to meet about once every two weeks. This may be adjusted when the new school year starts, but as of July 31, 2020 the group had already met three times since the end of the ADJOINT workshop.

- Continue honing skills in error analysis
- Write and publish a summary of Cuyt's CMA Group paper (Backeljauw et al., 2014) accessible to non-experts
- Design a template for presenting the error analysis for each function in DLMF Tables
- Study the C++ code for each function in DLMF Tables in order to
    - Facilitate code compilation at NIST
    - Produce codes for additional functions
- Assess the system from the point of view of different users
    - Novice user – undergraduate student, non-specialist
    - Researcher – mathematician, computer scientist, statistician, chemist, physicist, engineer
    - Numerical analyst – specialist in accuracy and stability of algorithms
    - Software developer – one with specialized software requirements, commercial developer

Impact: By virtue of its connection to the DLMF, the DLMF Tables project has high visibility within ACMD and ITL. The DLMF Tables project team currently has two components: NIST group, Antwerp group. I see the ADJOINT group as the beginning of a third component that can serve as a bridge between NIST and Antwerp. The NIST team is primarily responsible for the front-end interface to DLMF Tables, while Antwerp is responsible for the back-end computational engine. To create an efficient, coherent site, each group must have a clear understanding of the other's work. An accessible summary of the Backeljauw paper will help bridge the 'knowledge' gap between the two groups, and it can also be made available on the site to aid the understanding of users. Currently, DLMF Tables is located on a server at the University of Antwerp. The plan is to eventually move the main site to NIST, while possibly having a mirror site at Antwerp. As the transition occurs, I would like to see the NIST and Antwerp knowledge bases converge so that they look similar at both locations. I would be very happy to see the ADJOINT group continue its work on the project. In any case, I believe this has been an invaluable experience for all of us.

Additional Funding: Since I am a federal government employee at NIST, I cannot accept outside funding for this work. However, I am wondering if initial funding allotted for me can instead be provided to the other members of my group for books, travel for conferences, travel to NIST, etc.

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I sincerely thank the ADJOINT 2020 Program Directors and MSRI for a wonderful research experience. I was very impressed with the patience, kindness and attentiveness of all staff members when assisting us. I think the ADJOINT Workshop is an excellent idea and I believe it probably will lead to many lasting research collaborations. I will highly recommend it to young PhDs in the mathematical sciences. I look forward to getting together with many of the participants in the future.

Bonita Saunders