Dp-finite fields reading seminar
paper VI, §4,5 — part 2

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Recap

Theorem 4.10, 4.13

Let (K, τ) be W-topological. Let $\tau_1, \dots, \tau_n$ be the local components of τ.

1. Each τ_i has a unique V-topological coarsening τ_i^V.
2. Map $\tau_i \longmapsto \tau_i^V$ is bijection between local components of τ and V-topological coarsenings of τ.
3. τ is an independent sum of the τ_i.

Theorem (Theorem 4.13)

Let τ be a W-topology on a field K. Let $\tau_1, \ldots, \tau_n$ be the local components of τ. Then τ is an independent sum of the τ_i.

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Proof of 4.13

Fix an ultrapower K^* of K. Let $R, R_1, \ldots, R_n$ be the $\bigvee$-definable rings corresponding to $\tau, \tau_1, \ldots, \tau_n$. By the dictionary, the R_i are key localizations of R. For each i, let $S_i = R_1 \cap \ldots \cap R_i$. Note:

- $S_1 = R_1$ and $S_n = R$ (by Proposition 4.5),
- each S_i is a K-subalgebra of K^* which is $\bigvee$-definable over K, and each S_i is a W_n ring since $S_i \supseteq R$ (by Lemma 2.7 in paper V).

By dictionary, S_i corresponds to a W-topology σ_i. Note

- $\sigma_1 = \tau_1$ and $\sigma_n = \tau$.

The key localizations of S_i are $R_1, \ldots, R_i$, so the local components of σ_i are $\tau_1, \ldots, \tau_i$.
Consider the following claim....

Proof of 4.13

Claim

σ_i is an independent sum of σ_{i-1} and τ_i.

Proof of claim

First, to show independence of σ_{i-1} and τ_i. Let τ_j^V denote the unique V-topological coarsening of τ_j. By Theorem 4.10, the set of V-topological coarsenings of σ_{i-1} is

$$\{\tau_1^V, \dots, \tau_{i-1}^V\}.$$

Note that τ_i^V is not in this set! So σ_{i-1} and τ_i have no common V-topological coarsening. Since also they are both coarsenings of τ which is a W-topology, it follows that they are independent (by Theorem 7.16, paper V).

Proof of 4.13

Proof of claim

Second, to show that σ_{i-1} and τ_i generate σ_i. Let's work in K^*, where $\sigma^*_{i-1}, \tau^*_i, \sigma^*_i$ denote the topologies corresponding to S_{i-1}, R_i, S_i. By definition $S_i = S_{i-1} \cap R_i$. For any nonzero a there are nonzero b, c such that

$$bS_{i-1} \cap cR_i \subseteq aS_i.$$

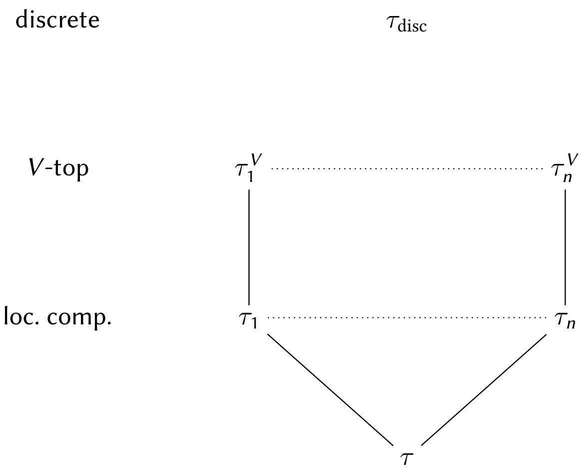
For $a = b = c$, equality! This proves

$$\forall U \in \sigma^*_i \exists V \in \sigma^*_{i-1} \exists W \in \tau^*_i \ : \ V \cap W \subseteq U.$$

Conversely,

$$\forall V \in \sigma^*_{i-1} \forall W \in \tau^*_i \exists U \in \sigma^*_i \ : \ U \subseteq V \cap W,$$

simply because we can take $U = V \cap W$. (Note $V, W \in \sigma^*_i$ since σ^*_i is finer than σ^*_{i-1} and τ^*_i.)
By properties of local sentences, both the above hold for $\sigma_{i-1}, \tau_i, \sigma_i$ in placed of their ultra-counterparts. Therefore σ_i is generated by σ_{i-1} and τ_i as required. $\square_{\text{claim}}$


discrete
τ_{disc}
V-top
τ_1^V
τ_n^V
loc. comp.
τ_1
τ_n
τ

VI, Corollary 4.15

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Let (K, τ) be W-topological.

1. If $\mathrm{char}(K) \neq 2$ and the squaring map $X^2 : K^\times \longrightarrow K^\times$ is an open map, then τ is local and has a unique V-topological coarsening.
2. If $\mathrm{char}(K) = p > 0$ and the Artin-Schreier map $\mathcal{P} : K \longrightarrow K$ is an open map, then τ is local and has a unique V-topological coarsening.

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Proof

Let $\tau_1, \ldots, \tau_n$ be local components of τ. Suppose $n > 1$, for a contradiction. By Thm.4.13, τ is independent sum of $\tau_1, \ldots, \tau_n$.

1. Suppose $\mathrm{char}(K) \neq 2$. (Cf V.6.9) Use 'weak approximation' to find $x \in K^*$ s.t.

- x infinitesimally close to 1 wrt τ_1, and
- x infinitesimally close to -1 wrt $\tau_2, \ldots, \tau_n$.

Thus x^2 is infinitesimally close to 1 wrt $\tau_1, \ldots, \tau_n$, hence wrt τ. But x is neither infinitesimally closed to ± 1 with respect to τ. Thus X^2 not τ-open map – $\perp$.

VI, Corollary 4.15

Let (K, τ) be W-topological.

1. If $\mathrm{char}(K) \neq 2$ and the squaring map $X^2 : K^\times \longrightarrow K^\times$ is an open map, then τ is local and has a unique V-topological coarsening.
2. If $\mathrm{char}(K) = p > 0$ and the Artin-Schreier map $\mathcal{P} : K \longrightarrow K$ is an open map, then τ is local and has a unique V-topological coarsening.

Proof (cont.)

2. Suppose $\mathrm{char}(K) = p > 0$. (Cf V.6.9) Use 'weak approximation' to find $x \in K^*$ s.t.

– x infinitesimally close to 0 wrt τ_1, and

– x infinitesimally close to 1 wrt $\tau_2, \ldots, \tau_n$.

Thus $\mathcal{P}(x)$ is infinitesimally close to 0 wrt $\tau_1, \ldots, \tau_n$, hence wrt τ. But x is infinitesimally close to no root of $\mathcal{P}$ in K. Thus $\mathcal{P}$ not τ-open – $\perp$.

towards VI, Corollary 4.16

Black box from II [Proposition 5.17]

K unstable dp-finite, with monster $\mathbb{K} \succ K$. Let I_K set of additive K-infinitesimals.

1. I_K is an additive subgroup of K.
2. $I_K = I_K \cdot I_K$, where $I_K \cdot I_K = \{\sum_{\text{finite}} x_i y_i \mid x_i, y_i \in I_K\}$.
3. $1 + I_K$ is multiplicative subgroup of $K^\times$, and $-1 \notin I_K$.
4. For every $n \geq 1$,
$$X^n : 1 + I_K \longrightarrow I_K$$
is surjective.
5. If $\mathrm{char}(K) = p > 0$, then
$$\mathcal{P} : I_K \longrightarrow I_K$$
is surjective.

VI, Corollary 4.16

$(K, +, \cdot, \ldots)$ is an expansion of a field, allowed to be trivial expansion.

VI, Corollary 4.16 (part a)

1. If $(K, +, \cdot, \ldots)$ unstable dp-finite, then the canonical topology is local W-topology.
2. If $(K, +, \cdot, \ldots)$ unstable dp-finite, then it admits a unique definable V-topology.

Proof

Let τ be canonical topology on K. By V.6.3, (K, τ) is W_n-topological, dp-rk $= n$. By Black Box

- For every $U \in 1 + \tau$ there exists $V \in 1 + \tau$ s.t. $V \subseteq U^{(2)}$.
- (Case $\mathrm{char}(K) = p > 0$) For every $U \in \tau$ there exists $V \in \tau$ s.t. $V \subseteq \mathcal{P}(U)$.

By 4.15, τ is local and has a unique V-topological coarsening. Done for 1.!
By V.6.9, V-topological coarsenings of τ are exactly the definable V-topologies. Done for 2.!

3. If $(K, +, \cdot, v)$ dp-finite valued field, then v is henselian.
4. If $(K, +, \cdot)$ dp-finite field, neither finite nor ACF nor RCF, then K admits a non-trivial definable henselian valuation.
5. The conjectural classification of dp-finite fields holds!

Let's take a look at IV.6.4.